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New optical soliton solutions of the popularized anti-cubic nonlinear Schrödinger equation versus its numerical treatment

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Abstract
In our current article, we will use two diverse methods namely the extended simple equation method (ESEM) and the extended direct algebraic method (EDAM) to extract the soliton solutions of popularized anti-cubic nonlinear Schrödinger equation that is very useful in the field of the optics. The obtained rational solutions via these two reliable, effective techniques denote the importance of these methods. Moreover, we will implement the differential transform method (DTM) which is one of the most, new semi-analytical and numerical methods to construct the corresponding numerical solutions for all achieved soliton solutions by the above two methods. We will compare between the soliton solutions introduced by the two suggested methods with the numerical solutions obtained by the DTM. It is clear that there exist similarity and convergence between the traveling wave solutions achieved by the ESEM, EDAM and the numerical solutions achieved by DTM. The novelty of our achieved solutions will appear when it compared by [1].

Keywords The nonlinear Schrödinger equation · Extended simple equation method · Extended direct algebraic method · Differential transform method (DTM) · Traveling wave solutions · Numerical solutions

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1 Introduction

Nonlinear Schrödinger equation (NLSE) is one of the most important and famous in vast range of nonlinear science and engineering fields. It is a classical field equation whose principal applications are the propagation of light in nonlinear optical fibers and planar waveguides (Zafar et al. 2021). Optical solitons along with the generalized anti-cubic non-linearity have been determined by the extended trial function method in Biswas et al. (2019). There are some trails to study various forms of the NLSE see for example, Biswas et al., (2017) who introduced the optical solitons of perturbation along with anti-cubic nonlinearity, new visions of the soliton solutions to the modified nonlinear Schrodinger equation were introduced via (Bekir and Zahran 2021a, 2021b) introduced new multiple-different impressive perceptions for the solitary wave solution to the magneto-optic waveguides with anti-cubic nonlinearity, the soliton solutions to the higher-order nonlinear Schrodinger equation have been obtained using three distinct and impressive visions in Bekir and Zahran (2021c), different wave structures to the generalized third-order nonlinear Schrödinger equation have been investigated in Hosseini et al. (2020), Bright and dark soliton solutions for the complex Kundu-Eckhaus equation were discussed in Bekir and Zahran (2020), comparison between the new exact and numerical solutions of the Mikhailov-Novikov-Wang equation has been revealed in Bekir et al. (2021).

In some problems, the analytical methods are mostly very hard and difficult to obtain the analytical solutions, for this reason it is desirable to use the semi-analytic and numerical methods to obtain the numerical solutions. One of the most, recent, effective semi-analytic and numerical methods is the differential transform method that was firstly introduced by Zhou (2010). The solution of the partial differential equations by two-dimensional differential transform method was considered by Chen & Ho in (1999). Ziyaee (2015) introduced the solution of two-dimensional Fredholm integral equation using differential transform method. Ibrahim (2022), represented the application of DTM & Adomian polynomial for solving RLC circuits problems and higher order differential equations. Kajani and Shehni (2011) used the DTM to solve nonlinear Volterra integro-differential equations and introduced the solutions of delay differential equations, (Karakoç and Bereketoglu 2009). For more details about the application of DTM to actual world problems, see (Patel and Dhodiya 2022; Khatib 2016).

In related subject, there are relevant articles that are useful in this field see for example Kaplan and Raza (2022) who found the complexion solutions of the Hirota–Satsuma–Ito equation and generalized Calogero–Bogoyavlenskii–Schiff equation by using the Hirota bilinear form and the extended transformed rational function method, Fadhal, et al. (2022) who obtained the travelling wave solutions of the Sasa-Satsuma equation by using the the modified simple equation and the exponential rational function approach, Kaabar, et al. (2021) who improved the sub equation method and the sine–cosine method to extract various types of solutions such as the hyperbolic function solutions, the trigonometric function solutions, and the rational solutions for the time-fractional generalized Fitzhugh–Nagumo equation with time-dependent coefficients involving the conformable fractional derivative, Raza, et al. (2022) who extracted the auto-Bäcklund transformations of $(4+1)$-dimensional Boiti–Leon–Manna–Pempinelli equation according to the extended homogeneous balance method and used these transformations to study analytic explicit solutions of this equation, derived the complexiton solutions of this model by using the Hirota bilinear form and the extended transformed rational function method. Moreover, there are some new methods that are recently used to solve the NLPDE see for example Wang, (2023a) who used the

Cole-Hopf transformation to construct the condition for finding the diverse soliton solutions of the Fokas system via the different trial functions, obtained some new soliton solutions such as the M-shaped rational, interaction, cross-periodic, double exponential form and multi-waves soliton solutions, Wang, (2022a) who used the direct mapping method to study the generalized third-order nonlinear Schrödinger's equation, constructed abundant solutions in only one step, thus avoided a lot of tedious calculations and Wang, (2023b) who used a fast algorithm namely Yang's special function method to find abundant non-differentiable exact solutions for the modified fractional Benjamin–Bona–Mahony equation. In same connection, there are many other different methods available for constructing the soliton solutions see for example Wang, (2022b) who used the variational direct method and He's frequency formulation method to seek the travelling wave solutions to the modified Benjamin-Bona-Mahony equation and obtained abundant exact solutions such as the bright wave, bright-dark wave, bright-like wave, kinky-bright wave and periodic wave solutions, Wang and Si, (2022) who derived abundant optical soliton solutions expressed in terms of generalized hyperbolic, generalized trigonometric, hyperbolic, and trigonometric functions of the Radhakrishnan–Kundu–Lakshmanan equation by using the Sardar sub-equation method and sub-equation method, Wang, (2022c) who derived a new fractal unsteady Korteweg–de Vries model which can model the shallow water with the non-smooth boundary, investigated the abundant exact solutions by means of the sub-equation method, Wang, et al. (2022) who used two powerful approaches namely the variational method and energy balance theory to search for the periodic wave solutions for the fractal generalized fourth-order Boussinesq equation and Wang, (2022d) who used the homogeneous balance method to construct the fractal Bäcklund transformation, employed it to extract some new exact explicit solutions such as the algebraic solitary wave solution of rational function, single-soliton solution, double-soliton solutions, N-soliton solutions, singular traveling solutions and the periodic wave solutions of trigonometric function to the fractal modification of the combined KdV–mKdV equation.

The popularized anti-cubic nonlinear Schrödinger equation is one of important models that describes the soliton propagation in optical fiber and plays effective role in improvement pulse propagation in optical fibers. The suggested NLSE (Zafar et al. 2021)

$$t\frac{\partial q}{\partial t} + a\frac{\partial^2 q}{\partial x^2} + b\frac{\partial^2 q}{\partial x \partial t} + \left\{c_1|q|^{-2(n+1)} + c_2|q|^{2n} + c_3|q|^{2(n+1)}\right\}q = 0 \tag{1}$$

where, $a, b, c_i, i = 1, 2, 3$ are constants, $q(x, t)$ is a wave profile depends on parametersx, t. Coefficients of $c_i, i = 1, 2, 3$ popularized anti-cubic nonlinearity along n being popularization parameter. If $c_1 = 0$, then there is no popularized term. This sequence of non-linearity and dispersion carries the required balance for the presence of wave solutions, (Zafar et al. 2021).

1.1 Characterization of the model of the problem

Consider the nonlinear PDE (1) in which consider the successive transform

$$q(x, t) = \mathrm{V}(\zeta) \times \exp(\mathrm{i}\varphi), \zeta(x, t) = x - \tau t, \varphi(x, t) = -\mathrm{k}x + \omega t + \theta \tag{2}$$

In which $\mathrm{V}(\zeta)$ is real function which describes the geometry of the wave, τ is soliton velocity with phase (x, t). k is wave frequency, ω is wave number, θ is phase constant. By using the above mentioned transformation (2) and its required derivatives we approach to the real part as

$$(a - b\tau)V^{(2n+1)}V'' + \left(bk\omega - \omega - ak^2\right)V^{(2n+2)} + c_1 + c_2V^{(4n+2)} + c_3V^{(4n+4)} = 0 \tag{3}$$

While the imaginary part is

$$(b\omega - 2ak + \tau(bk - 1))V^{(2n+1)}V' = 0. \tag{4}$$

Equation (4) introduces the soliton's velocity in the form

$$\tau = \frac{b\omega - 2ak}{1 - bk}. \tag{5}$$

For which $bk \neq 1$. To obtain the solutions, we apply the following transformation

$$\mathrm{V}(\zeta) = [\mathrm{R}(\zeta)]^{\frac{1}{n+1}} \tag{6}$$

where $R(\zeta)$ is also a new function of ζ. By substituting from Eq. (6) into Eq. (3), it becomes

$$\left[a(bk+1) - \omega b^2\right]\left[(n+1)RR'' - n\left(R'\right)^2\right] + \frac{(n+1)}{1-bk}\left[c_1 + \left(bk\omega - \omega - ak^2\right)R^2 + c_3R^4 + c_2R^{\frac{4n+2}{n+1}}\right] = 0. \tag{7}$$

For fulfillment, put $c_2 = 0$. This leads to the retraining of the representative of study given below

$$\imath\frac{\partial q}{\partial t} + a\frac{\partial^2 q}{\partial x^2} + b\frac{\partial^2 q}{\partial x\partial t} + \left\{c_1|q|^{-(2n+2)} + c_3|q|^{(2n+2)}\right\}q = 0 \tag{8}$$

Similarly, Eq. (17) will be reduced to

$$\left[a(bk+1) - \omega b^2\right]\left[(n+1)RR'' - n\left(R'\right)^2\right] + \frac{(n+1)}{1-bk}\left[c_1 + \left(bk\omega - \omega - ak^2\right)R^2 + c_3R^4\right] = 0. \tag{9}$$

2 The analytical solution methods

In our current work we will introduce two of the most important and an effective analytical method, the first one is the ESEM while the second is the EDAM for solving the suggested model Eq. (9).

2.1 The structure of ESEM

To investigate the solution of any NLPDE using ESEM, let us propose the general form of the any NLPDE

$$U\left(\phi, \phi_x, \phi_t, \phi_{xx}, \phi_{tt}, \dots\right) = 0. \tag{10}$$

where U in terms of the successive partial differentiation to $\phi(x,t)$, when Eq. (10) surrenders to the transformation $\phi(x,t) = \varphi(\zeta), \zeta = x - \tau t$ it will be transformed to ordinary differential equations ODE

$$R\left(\varphi, \varphi', \varphi'', \varphi''', \dots\right) = 0. \tag{11}$$

where R in terms of the total derivatives $\varphi(\zeta)$.

The solution from the point of view of the ESEM is

$$R(\zeta) = \sum_{i=-M}^{M} A_i \varphi^i(\zeta) \tag{12}$$

where $\varphi(\zeta)$ will be determined from

$$\varphi^{'}(\zeta) = a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3 \tag{13}$$

The positive integers M that appears in Eq. (12) can be determined by the homogeneous balance, while A_i are constants that will be determined later, and arbitrary constants a_0, a_1 and a_2 will generate these cases.

(1) If $a_1 = a_3 = 0$ it will be transformed to the Riccati equation, which has the following solutions

$$\varphi(\zeta) = \frac{\sqrt{a_0 a_2}}{a_2}\tan\left(\sqrt{a_0 a_2}(\zeta + \zeta_0)\right), a_0 a_2 > 0 \tag{14}$$

$$\varphi(\zeta) = \frac{\sqrt{-a_0 a_2}}{a_2}\tanh\left(\sqrt{-a_0 a_2}\zeta - \frac{\rho_1 \ln\zeta_0}{2}\right), a_0 a_2 < 0, \zeta > 0, \rho_1 = \pm 1 \tag{15}$$

(2) If $a_0 = a_3 = 0$, it will be transformed to the Bernoulli equation, which has the following solutions

$$\varphi(\zeta) = \frac{a_1 \mathrm{Exp}\left[a_1\left(\zeta + \zeta_0\right)\right]}{1 - a_2 \mathrm{Exp}\left[a_1\left(\zeta + \zeta_0\right)\right]}, a_1 > 0 \tag{16}$$

$$\varphi(\zeta) = \frac{-a_1 \mathrm{Exp}\left[a_1\left(\zeta + \zeta_0\right)\right]}{1 + a_2 \mathrm{Exp}\left[a_1\left(\zeta + \zeta_0\right)\right]}, a_1 < 0 \tag{17}$$

And the general solution to ansatz Eq. (11) is

$$\varphi(\zeta) = -\frac{1}{a_2}\left(a_1 - \sqrt{4a_1a_2 - a_1^2}\tan\left(\frac{\sqrt{4a_1a_2 - a_1^2}}{2}\left(\zeta + \zeta_0\right)\right)\right), 4a_1a_2 > a_1^2, a_2 > 0, \tag{18}$$

$$\varphi(\zeta) = \frac{1}{a_2}\left(a_1 + \sqrt{4a_1a_2 - a_1^2}\tanh\left(\frac{\sqrt{4a_1a_2 - a_1^2}}{2}\left(\zeta + \zeta_0\right)\right)\right), 4a_1a_2 > a_1^2, a_2 < 0. \tag{19}$$

where ζ_0 is the constant of integration.

By inserting Eq. (13) into Eq. (12) and implementing the equivalence for various powers of ϕ^i this emerges a system of unknowns by solving it these parameters will be extracted.

Now we apply this method on Eq. (9) for which the homogeneous balance can be calculated by balancing $RR'' andR^4$ that implies $M = 1$ then the solution can be written in the form

$$R(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1\varphi \tag{20}$$

where $\varphi'(\zeta) = a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3$

Case 1 The first family in which $a_1 = a_3 = 0 \Rightarrow \varphi' = a_0 + a_2\varphi^2$, consequently

$$R' = -\frac{a_0 A_{-1}}{\phi^2} + \left(A_1 a_0 - a_2 A_{-1}\right) + a_2 A_1 \phi^2 \tag{21}$$

$$R'' = \frac{2a_0^2 A_{-1}}{\phi^3} + \frac{2a_0 a_2 A_{-1}}{\phi} + 2a_0 a_2 A_1 \phi + 2a_2^2 A_1 \phi^3 \tag{22}$$

By substituting from the relations (20–22) into Eq. (9) and accomplishing the equivalence for various powers of φ^i, this emerges system of equations by solving it we get.

$$\begin{aligned}
&(1)\ c_3 \to -\frac{(-1+bk)(2+n)\omega a_2^2}{\left(k^2(1+n) - 4\left(-1+b^2k^2\right)a_0a_2\right)A_1^2}, c_1 \to 0, A_0 \to 0, A_{-1} \to \frac{a_0A_1}{a_2},\\
&\quad a \to \frac{(-1+bk)\omega\left(-1-n+4b^2a_0a_2\right)}{-k^2(1+n)+4\left(-1+b^2k^2\right)a_0a_2}.\\
&(2)\ c_3 \to -\frac{(-1+bk)(2+n)\omega a_2^2}{\left(k^2(1+n)+2\left(-1+b^2k^2\right)a_0a_2\right)A_1^2}, c_1 \to \frac{(-1+bk)n\omega a_0^2A_1^2}{k^2(1+n)+2\left(-1+b^2k^2\right)a_0a_2}, A_0 \to 0, A_{-1} \to 0,\\
&\quad a \to \frac{(-1+bk)\omega(1+n+2b^2a_0a_2)}{k^2(1+n)+2(-1+b^2k^2)a_0a_2}\\
&(3)\ c_3 \to -\frac{(-1+bk)(2+n)\omega a_2^2}{\left(k^2(1+n)+8\left(-1+b^2k^2\right)a_0a_2\right)A_1^2}, c_1 \to \frac{16(-1+bk)n\omega a_0^2A_1^2}{k^2(1+n)+8\left(-1+b^2k^2\right)a_0a_2}, A_0 \to 0,\\
&\quad A_{-1} \to -\frac{a_0A_1}{a_2}, a \to \frac{(-1+bk)\omega(1+n+8b^2a_0a_2)}{k^2(1+n)+8(-1+b^2k^2)a_0a_2}\\
&(4)\ c_3 \to -\frac{(-1+bk)(2+n)\omega a_0^2}{\left(k^2(1+n)+2\left(-1+b^2k^2\right)a_0a_2\right)A_{-1}^2}, c_1 \to \frac{(-1+bk)n\omega a_2^2A_{-1}^2}{k^2(1+n)+2\left(-1+b^2k^2\right)a_0a_2}, A_0 \to 0, A_1 \to 0,\\
&\quad a \to \frac{(-1+bk)\omega(1+n+2b^2a_0a_2)}{k^2(1+n)+2(-1+b^2k^2)a_0a_2}\\
&(5)\ c_3 \to -\frac{2(-1+bk)(2+n)\omega a_2^2}{3k^2(1+n)A_1^2}, c_1 \to 0, A_0 \to 0, a_0 \to -\frac{k^2(1+n)}{8\left(-1+b^2k^2\right)a_2},\\
&\quad A_{-1} \to -\frac{k^2(1+n)A_1}{8(-1+b^2k^2)a_2^2}, a \to \frac{(-2+3b^2k^2)\omega}{3k^2(1+bk)}\\
&(6)\ c_3 \to -\frac{(-1+bk)(2+n)\omega a_2^2}{3k^2(1+n)A_1^2}, c_1 \to \frac{k^2n(1+n)\omega A_1^2}{3(-1+bk)(1+bk)^2a_2^2}, A_0 \to 0,\\
&\quad a_0 \to \frac{k^2(1+n)}{4(-1+b^2k^2)a_2}, A_{-1} \to -\frac{k^2(1+n)A_1}{4(-1+b^2k^2)a_2^2}, a \to \frac{(-1+3b^2k^2)\omega}{3k^2(1+bk)}
\end{aligned} \tag{23}$$

From which we will implement the first solution.

$$c_3 \to -\frac{(-1+bk)(2+n)\omega a_2^2}{\left(k^2(1+n)-4\left(-1+b^2k^2\right)a_0a_2\right)A_1^2}, c_1 \to 0, A_0 \to 0, A_{-1} \to \frac{a_0A_1}{a_2},$$

$$a \to \frac{(-1+bk)\omega\left(-1-n+4b^2a_0a_2\right)}{-k^2(1+n)+4\left(-1+b^2k^2\right)a_0a_2}.$$

This result can be simplified to be

$$c_3 = \omega = A_{-1} = n = k = 1\&b = A_1 = -1 \tag{24}$$

Hence,

$$a_0 = \frac{-1}{\sqrt{3}}\&a_2 = \frac{1}{\sqrt{3}}\&a = \frac{-10}{3}\text{OR}a_0 = \frac{1}{\sqrt{3}}\&a_2 = \frac{-1}{\sqrt{3}}\&a = \frac{-10}{3} \tag{25}$$

So the solution according to these values, since $a_0a_2 \prec 0, a_0 = \frac{-1}{\sqrt{3}}\&a_2 = \frac{1}{\sqrt{3}}$, is

$$\varphi(\zeta) = \frac{\sqrt{-a_0a_2}}{a_2}\tanh\left(\sqrt{-a_0a_2}\zeta - \frac{\rho_1\ln\zeta_0}{2}\right), a_0a_2 \prec 0, \zeta_0 = 2, \rho_1 = -1$$

$$\varphi(\zeta) = \tanh\left(\frac{1}{\sqrt{3}}\zeta + 0.3\right) \tag{26}$$

Then the solution of Eq. (9) is

$$R(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1\varphi = \coth\left(\frac{1}{\sqrt{3}}\zeta + 0.3\right) - \tanh\left(\frac{1}{\sqrt{3}}\zeta + 0.3\right) \tag{27}$$

Case 2 The first family in which $a_0 = a_3 = 0 \Rightarrow \varphi' = a_1\varphi + a_2\varphi^2$, consequently.

$$R' = \frac{-a_1A_{-1}}{\varphi} - a_2A_{-1} + A_1a_1\varphi + a_2A_1\varphi^2 \tag{28}$$

$$R'' = \frac{a_1^2A_{-1}}{\varphi} + a_1a_2A_{-1} + a_1^2A_1\varphi + 3a_1a_2A_1\varphi^2 + 2a_2^2A_1\varphi^3. \tag{29}$$

By substituting from the relations (20) and (28–29) into Eq. (9) and accomplishing the equivalence for various powers of φ^i, this emerges system of equations by solving it we get following possible solutions.

$$c_3 \to \frac{(2+n)\left(\text{ak}^2+2\omega\right)}{2A_0^2}, c_1 \to -\frac{1}{2}n\left(\text{ak}^2+2\omega\right)A_0^2, b \to -\frac{1}{k},$$

$$A_1 \to -\frac{2\sqrt{\omega}a_2A_0}{k\sqrt{1+n}\sqrt{\mathrm{ak}^2+2\omega}}, a_1 \to -\frac{k\sqrt{1+n}\sqrt{\mathrm{ak}^2+2\omega}}{\sqrt{\omega}}, A_{-1} \to 0$$

$$c_3 \to \frac{(2+n)\left(\mathrm{ak}^2+\omega-bk\omega\right)}{2A_0^2}, c_1 \to -\frac{1}{2}n\left(\mathrm{ak}^2+\omega-bk\omega\right)A_0^2, A_1 \to \frac{2a_2A_0}{a_1},$$

$$A_{-1} \to 0, a \to \frac{2(1+n)(\mathrm{ak}^2+\omega-bk\omega)+b^2(-1+bk)\omega a_1^2}{(-1+b^2k^2)a_1^2} \tag{30}$$

$$c_3 \to \frac{(2+n)\left(\mathrm{ak}^2+2\omega\right)}{2A_0^2}, c_1 \to -\frac{1}{2}n\left(\mathrm{ak}^2+2\omega\right)A_0^2, b \to -\frac{1}{k},$$

$$A_1 \to \frac{2\sqrt{\omega}a_2A_0}{k\sqrt{1+n}\sqrt{\mathrm{ak}^2+2\omega}}, a_1 \to \frac{k\sqrt{1+n}\sqrt{\mathrm{ak}^2+2\omega}}{\sqrt{\omega}}, A_{-1} \to 0$$

From which we will take the first solution for illustration

$$c_3 \to \frac{(2+n)\left(\mathrm{ak}^2+2\omega\right)}{2A_0^2}, c_1 \to -\frac{1}{2}n\left(\mathrm{ak}^2+2\omega\right)A_0^2, b \to -\frac{1}{k},$$

$$A_1 \to -\frac{2\sqrt{\omega}a_2A_0}{k\sqrt{1+n}\sqrt{\mathrm{ak}^2+2\omega}}, a_1 \to -\frac{k\sqrt{1+n}\sqrt{\mathrm{ak}^2+2\omega}}{\sqrt{\omega}}, A_{-1} \to 0$$

This result can be simplified to be

$$\omega = A_0 = n = k = A_1 = 1\&b = -1\&a = 0 \tag{31}$$

Hence,

$$c_1 = -1\&c_3 = 3\&a_1 = -2\&a_2 = -1 \tag{32}$$

Since $a_1 = -2 < 1$ then according to Eq. (17)

$$\varphi(\zeta) = \frac{-a_1\mathrm{Exp}\left[a_1\left(\zeta+\zeta_0\right)\right]}{1+a_2\mathrm{Exp}\left[a_1\left(\zeta+\zeta_0\right)\right]} = \frac{2\mathrm{Exp}\left[-2\left(\zeta+\zeta_0\right)\right]}{1-\mathrm{Exp}\left[-2\left(\zeta+\zeta_0\right)\right]} = \frac{2\mathrm{Exp}[-2(\zeta+1)]}{1-\mathrm{Exp}[-2(\zeta+1)]}, \zeta_0 = 1 \tag{33}$$

Then the solution of Eq. (9) is

$$R(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1\varphi = 1 + \frac{2\mathrm{Exp}[-2(\zeta+1)]}{1-\mathrm{Exp}[-2(\zeta+1)]} \tag{34}$$

2.2 The structure of the EDAM

According to this algorithm, the solution of Eq. (11) is

$$R(\zeta) = \sum_{i=0}^{M} b_i \varphi^i(\zeta), \varphi'^2 = \alpha\varphi^2 + \beta\varphi^3 + \gamma\varphi^4. \tag{35}$$

Now, we will apply this algorithm for the suggested model Eq. (9) mentioned above for which the equivalence of balance is M = 1, so the solution takes the form

$$R(\zeta) = b_0 + b_1\varphi. \tag{36}$$

Hence

$$R' = b_1\varphi' = b_1\left(\sqrt{\alpha\varphi^2 + \beta\varphi^3 + \gamma\varphi^4}\right) \tag{37}$$

$$R'' = b_1\varphi'' = b_1\left(\alpha\varphi + 1.5\beta\varphi^2 + 2\gamma\varphi^3\right) \tag{38}$$

$$\varphi'^2 = \alpha\varphi^2 + \beta\varphi^3 + \gamma\varphi^4 \tag{39}$$

$$\varphi'' = \alpha\varphi + 1.5\beta\varphi^2 + 2\gamma\varphi^3 \tag{40}$$

Bysubstitutingfrom the relations (36–40) into Eq. (9) and implementing the equivalence of the coefficients of different powers φ^i, this implies a system of equations whose solution is.

$$\alpha \to -\frac{1.(1.+n)\left(ak^2+\omega-1.bk\omega\right)}{(-1.+bk)\left(a+abk-1.b^2\omega\right)}, \beta \to 0., b_0 \to 0., c_1 \to 0.,$$

$$c_3 \to -\frac{1.(-1.+bk)(2.+n)\gamma(a(-1.-1.bk)+b^2\omega)}{(1.+n)b_1^2} \tag{41}$$

$$\beta \to 0., \gamma \to 0., b_1 \to 0., c_1 \to -0.25\left(a\left(-1.\alpha+k^2\left(-2.+b^2\alpha\right)\right)+\left(-2.+2.bk+b^2\alpha-1.b^3k\alpha\right)\omega\right)b_0^2,$$

$$c_3 \to \frac{a(-0.25\alpha+k^2(0.5+0.25b^2\alpha))+(0.5-0.5bk+0.25b^2\alpha-0.25b^3k\alpha)\omega}{b_0^2} \tag{42}$$

From which we will take the first solution for illustration

$$\alpha \to -\frac{1.(1.+n)\left(ak^2+\omega-1.bk\omega\right)}{(-1.+bk)\left(a+abk-1.b^2\omega\right)}, \beta \to 0., b_0 \to 0., c_1 \to 0.,$$

$$c_3 \to -\frac{1.(-1.+bk)(2.+n)\gamma(a(-1.-1.bk)+b^2\omega)}{(1.+n)b_1^2}$$

This result can be simplified to be

$$\omega = c_3 = n = k = a = 1 \& b = -1 \tag{43}$$

Hence,

$$\alpha = -3\&c_3 = 1\&\gamma = \frac{1}{3}\&b_1 = 1 \tag{44}$$

Consequently, from (39) we have

$$\varphi'(\zeta) = \sqrt{\alpha\varphi^2 + \beta\varphi^3 + \gamma\varphi^4} = \sqrt{-3\varphi^2 + \frac{1}{3}\varphi^4} = \sqrt{\frac{\varphi^4}{3} - 3\varphi^2}$$

$$\rightarrow \frac{d\varphi}{d\zeta} = \frac{\varphi}{\sqrt{3}}\sqrt{\varphi^2 - 9} \rightarrow \frac{d\varphi}{\varphi\sqrt{\varphi^2 - 9}} = \frac{d\zeta}{\sqrt{3}} \rightarrow \frac{1}{3}\sec^{-1}\left(\frac{\varphi}{3}\right) = \frac{\zeta}{\sqrt{3}} \rightarrow \varphi(\zeta) = 3\sec\left(\sqrt{3}\zeta\right) \tag{45}$$

$$\therefore R(\zeta) = b_0 + b_1\varphi(\zeta) = 3\sec\left(\sqrt{3}\zeta\right) \tag{46}$$

3 The Numerical treatment of NLSE using DTM

The NLSE appears in many fields such as physics and engineering. In almost problems the analytical methods are very hard and difficult and some of these problems have no exact solutions, so in this case it is desirable to use the semi-analytic and numerical methods. One of the most recent and effective semi-analytic and numerical methods is the differential transform method (DTM).

The differential transform of the k^{th} derivative of function $y(x)$ is defined as

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \tag{47}$$

where $Y(k)$ is the transformed function and $y(x)$ is the original function. Differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \approx y_N(x) = \sum_{k=0}^{N} Y(k)x^k \tag{48}$$

By substituting Eq. (47) in (48) we get

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{d^k y(x)}{dx^k}\bigg|_{x=0} \tag{49}$$

It indicates that the differential transform concept is derived by Taylor series expansion. In the previous definition we consider the case when $x = 0$, but it is true for any fixed real number $x = x_0$. The main theorems that can be derived from Eqs. (47) and (48) can be summarized in table (1) see (Ibrahim 2022) and (Khatib 2016).

Original functions	Transformed functions
$y(x) = u(x) \pm m(x)$	$Y(k) = U(k) \pm M(k)$
$y(x) = \alpha m(x)$	$Y(k) = \alpha M(k)$
$y(x) = \frac{du(x)}{dx}$	$Y(k) = (k+1)U(k+1)$

Original functions	Transformed functions
$y(x) = \frac{d^2u(x)}{dx^2}$	$Y(k) = (k+1)(k+2)U(k+2)$
$y(x) = \frac{d^v u(x)}{dx^v}$	$Y(k) = (k+1)(k+2)\mathrm{K}(k+n)U(k+n)$
$y(x) = x^m$	$Y(k) = \delta(k-m) = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$
$y(x) = g(x)h(x)$	$\sum_{m=0}^{k} H(m)G(k-m)$
$y(x) = e^{\lambda x}$	$Y(k) = \frac{\lambda^k}{k!}$
$y(x) = (1+x)^m$	$Y(k) = \frac{m(m-1)\ldots(m-k+1)}{k!}$
$f(y) = y^m$	$F(k) = \begin{cases} Y^m(0) & ifk = 0, \\ \frac{1}{Y(0)} \sum_{r=1}^{k} \frac{(m+1)r-k}{k} Y(r)F(k-r) & ifk \geq 1. \end{cases}$
$f(y) = e^{ay}$	$F(k) = \begin{cases} e^{aY(0)} & ifk = 0 \\ a \sum_{r=0}^{k-1} \frac{r+1}{k} Y(r+1)F(k-r-1) & ifk \geq 1 \end{cases}$
$f(y) = sin(ay)$	$F(k) = \begin{cases} sin(aY(0)) & \text{if } k = 0 \\ a \sum_{r=0}^{k-1} \frac{k-r}{k} G(r)Y(k-r) & \text{if } k \geq 1 \end{cases}$

3.1 DTM Solution Corresponding to ESEM (first family)

Consider Eq. (9) with the values

$$c_3 = \omega = A_{-1} = n = k = 1\&b = -1\&c_1 = 0\&a = \frac{-10}{3}$$

That becomes

$$-2RR'' + \left(R'\right)^2 + \frac{4}{3}R^2 + R^4 = 0. \tag{50}$$

With initial condition obtained from Eq. (27)

$$R(0) = 3.314143\&R'(0) = -6.75432 \tag{51}$$

The solutions of (50–51) is written as

$$R(\zeta) = \sum_{k=0}^{\infty} Y(k)\zeta^k \approx R_N(\zeta) = \sum_{k=0}^{N} Y(k)\zeta^k \tag{52}$$

where $Y(k)$ is the transformed function of $R(\zeta)$, which can be computed as follow. Using the theorems in table (1) in Eq. (50–51) we get

$$-2\sum_{r=0}^{k}(r+1)(r+2)Y(r+2)(Y(k-r)+\sum_{r=0}^{k}(r+1)(k-r+1)Y(r+1)(Y(k-r+1)$$
$$+\frac{4}{3}\mathrm{F(k)}+\mathrm{H(k)}=0 \tag{53}$$

where

$$\mathrm{F(k)}=\begin{cases} Y^2(0);k=0 \\ \frac{1}{Y(0)}\sum_{r=1}^{k}\frac{3r-k}{k}Y(r)F(k-r);k\geq 1\end{cases} \tag{54}$$

$$H(\mathrm{k})=\begin{cases} Y^4(0);k=0 \\ \frac{1}{Y(0)}\sum_{r=1}^{k}\frac{5r-k}{k}Y(r)H(k-r);k\geq 1\end{cases} \tag{55}$$

3.1.1 Putting $\mathbf{k=0, to find Y(2)}$

$$-2(1)(2)Y(2)Y(0)+1(1)Y(1)Y(1)+\frac{4}{3}Y^2(0)+Y^4(0)=0$$

where $Y(0)=R(0)=3.14143\&Y(1)=R'(0)=-6.75432$ then

$$Y(2)=\frac{Y(1)Y(1)+\frac{4}{3}Y^2(0)+Y^4(0)}{4Y(0)}=12.4280875 \tag{56}$$

By the same way we can get the remainder coefficients

$$Y(3)=-23.719481;Y(4)=45.707944;Y(5)=-88.0004270620603;Y[6]=169.3535766546133; \tag{57}$$

Substituting from Eq. (56–57) into Eq. (52) we get the solution of Eq. (50) in the form

$$R(\zeta)=\sum_{k=0}^{6}Y(k)\zeta^k=3.14143-6.75432\zeta+12.4280875\zeta^2-23.719481\zeta^3$$
$$+45.707944\zeta^4-88.0004270620603\zeta^5+169.3535766546133\zeta^6 \tag{58}$$

That is identical with the Taylor (Maclaurin) expansion of the exact solution (27) of Eq. (50)

$$R(\zeta)=Seires\left(\coth\left(\frac{1}{\sqrt{3}}\zeta+0.3\right)-\tanh\left(\frac{1}{\sqrt{3}}\zeta+0.3\right),\{\zeta,0,6\}\right)$$
$$=3.14143-6.75432\zeta+12.42806292\zeta^2-23.71943159\zeta^3+45.70783457\zeta^4$$
$$-88.00045437\zeta^5+169.3535473\zeta^6 \tag{59}$$

3.2 DTM Solution Corresponding to ESEM (Second family)

Consider Eq. (9) with the values

$$c_3 = 3; c_1 = -1; \omega = A_1 = A_0 = n = k = 1; b = -1; a = 0;$$

That becomes

$$-2RR'' + \left(R'\right)^2 - 2R^2 + 3R^4 - 1 = 0. \tag{60}$$

With initial condition obtained from Eq. (34)

$$R(0) = 1.31304 \& R'(0) = -0.724062 \tag{61}$$

The solutions of (60–61) is written as Eq. (52), where $Y(k)$ is the transformed function for $R(\zeta)$, which can be computed as follow.

By using the rules of DTM in Eq. (60–61) we get

$$\begin{aligned} &-2\sum_{r=0}^{k}(r+1)(r+2)Y(r+2)(Y(k-r) + \sum_{r=0}^{k}(r+1)(k-r+1)Y(r+1)(Y(k-r+1) \\ &-2\mathrm{F(k)} + 3\mathrm{H(k)} - \delta(\mathrm{k}) = 0 \end{aligned} \tag{62}$$

Where, F(k)&H(k) given from Eq. (54) and Eq. (55) respectively and

$$\delta(\mathrm{k}) = \begin{cases} 1; k = 0 \\ 0; k \geq 1 \end{cases} \tag{63}$$

Putting $\mathrm{k} = 0, 1, 2, 3, 4, \ldots$ to find the coefficients $\mathrm{Y(k)}, \mathrm{k} \geq 2$, with

$$Y(0) = R(0) = 1.31304 \& Y(1) = R'(0) = -0.724062$$

Then the coefficients are

$$Y(2) = 0.950719; Y(3) = -1.00697; Y(4) = 1.00528;$$

$$Y(5) = -1.00041; Y[6] = 0.999602; \tag{64}$$

Inserting the relations Eq. (61, 63, 64) and Eq. (54–55) into Eq. (62), we get the solution of Eq. (60) in the form

$$\begin{aligned} R(\zeta) = \sum_{k=0}^{6} Y(k)\zeta^k &= 1.31304 - 0.724062\zeta + 0.950719\zeta^2 - 1.00697\zeta^3 \\ &+ 1.00528\zeta^4 - 1.00041\zeta^5 + 0.999602\zeta^6 \end{aligned} \tag{65}$$

That is very identical with the Taylor (Maclaurin) expansion of the exact solution (34) of Eq. (60)

$$\begin{aligned} R(\zeta) &= Seires\left(1 + \frac{2 * \mathrm{e}^{-2*(\xi+1)}}{1 - \mathrm{e}^{-2*(\xi+1)}}\right) \\ &= 1.31304 - 0.724062\zeta + 0.950719\zeta^2 - 1.00697\zeta^3 \\ &\quad + 1.00528\zeta^4 - 1.00041\zeta^5 + 0.999602\zeta^6 + \ldots \end{aligned} \tag{66}$$

3.3 DTM Solution Corresponding to EDAM

Consider Eq. (9) with the values

$$c_1 = 0;\omega = c_3 = a = n = k = 1;b = -1;$$

That becomes

$$-2RR'' + \left(R'\right)^2 - 3R^2 + R^4 = 0. \tag{67}$$

With initial condition obtained from Eq. (46)

$$R(0) = 3\&R'(0) = 0 \tag{68}$$

The solutions of (67–68) is written as Eq. (52), where $Y(k)$ is the transformed function of $R(\zeta)$, which can be computed as follow.

By using the theorems in table (1) into Eq. (69–70) we get

$$\begin{aligned} &-2\sum_{r=0}^{k}(r+1)(r+2)Y(r+2)(Y(k-r)+\sum_{r=0}^{k}(r+1)(k-r+1)Y(r+1)(Y(k-r+1)\\ &-3\mathrm{F(k)}+\mathrm{H(k)}=0 \end{aligned} \tag{69}$$

where F(k)&H(k) given from Eq. (54) and Eq. (55) respectively.

Putting $\mathrm{k} = 0, 1, 2, 3, 4, \ldots \mathrm{tofindthecoefficientsY(k)}, \mathrm{k} \geq 2$, with

$$Y(0) = R(0) = 3\&Y(1) = R^{'}(0) = 0$$

Then the coefficients Y(k) obtained from (69) are

$$\begin{aligned} Y(2) &= \frac{9}{2};Y(3) = 0;Y(4) = \frac{45}{8};Y(5) = 0;Y[6] = \frac{549}{80};Y[7] = 0;Y[8]\\ &= \frac{7479}{896};Y[9] = 0;Y[10] = \frac{454689}{44800} \end{aligned} \tag{70}$$

Inserting the relations Eqs. (70) & (68) into Eq. (52) we get the solution of Eq. (67) in the form

$$R(\zeta) = \sum_{k=0}^{10} Y(k)\zeta^k = 3 + \frac{9}{2}\zeta^2 + \frac{45}{8}\zeta^4 + \frac{549}{80}\zeta^6 + \frac{7479}{896}\zeta^8 + \frac{454689}{44800}\zeta^{10} \tag{71}$$

That is very identical with the Maclaurin expansion of the exact solution (34) of Eq. (60)

$$R(\zeta) = Seires\left(3\sec\left(\sqrt{3}\zeta\right)\right) = 3 + \frac{9}{2}\zeta^2 + \frac{45}{8}\zeta^4 + \frac{549}{80}\zeta^6 + \frac{7479}{896}\zeta^8 + \frac{454689}{44800}\zeta^{10} \tag{72}$$

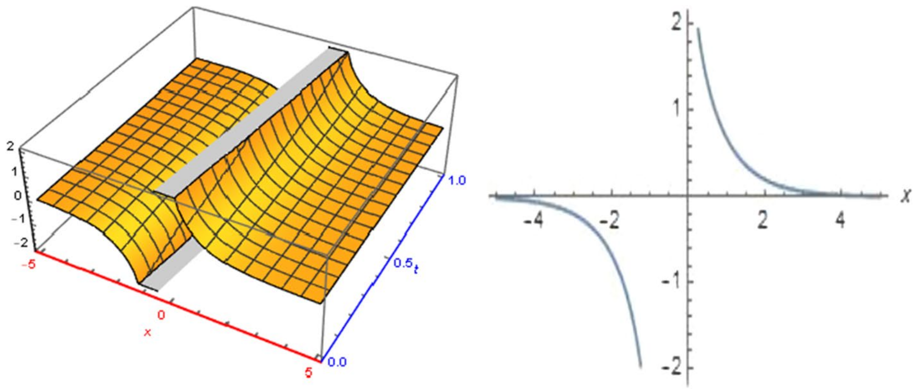

Fig. 1 This figure shows the solution Eq. (27) in 2D & 3D with values given in (24)

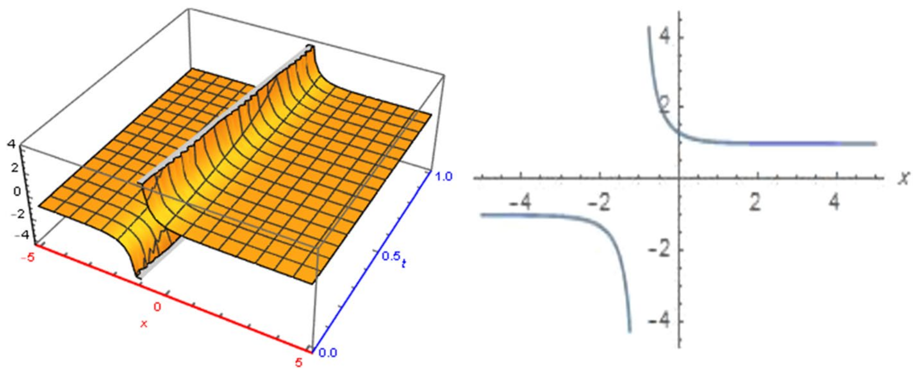

Fig. 2 This figure shows the solution of Eq. (34) in 2D & 3D with values given in (32)

4 Conclusion

In this research, we introduced two analytical solution methods of the NLSE as well as one numerical method to construct the new optical soliton solutions and its corresponding numerical solutions of the suggested model. The obtained free optical soliton solutions exactly using two interesting methods, the first one is the ESEM that gives the new optical soliton solution as in the Eq. (27) for the first case and Eq. (34) for the second case. Moreover, other new forms of the optical soliton solutions have been constructed by the EDAM as in the Eq. (46). The rational solutions that were obtained via these two reliable techniques are the parabolic soliton solution Fig. 1, exponential soliton solution Fig. 2, periodic soliton solution Fig. 3 that denote the effective and importance of these methods. The novelty of our obtained solutions will appear when it compared with (Zafar et al. 2021) who solved this model by modified extended tanh expansion method. Also, we used the DTM to extract the numerical solutions that are identical for all achieved optical soliton solutions by the two analytical methods. we compare the numerical results obtained using the DTM solution (58) by the Taylor (McLaurin) expansion of the first solution obtained by the ESEM first case Eq. (27) that shows the

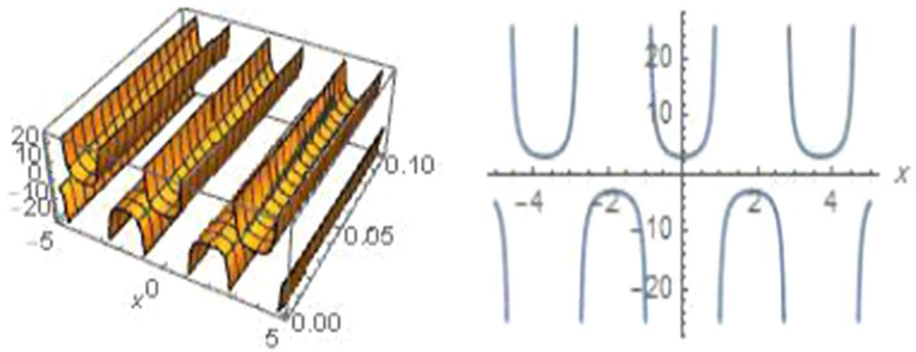

Fig. 3 This figure shows the solution of Eq. (46) in 2D & 3D with values given in (43)

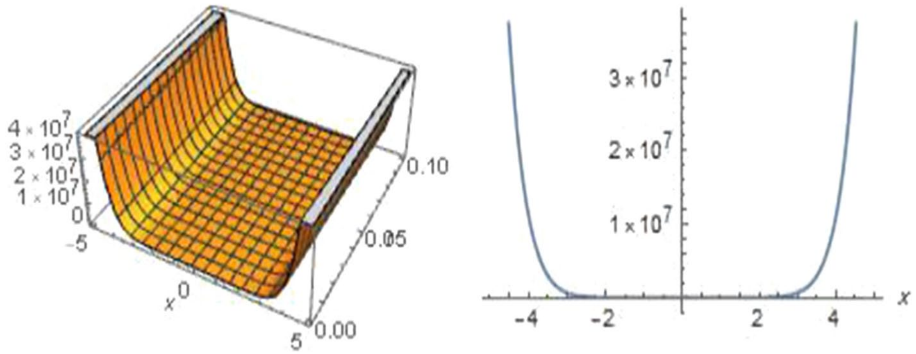

Fig. 4 This figure shows the solution of Eq. (71) in 2D & 3D using DTM

similarity between the exact and numerical solutions, also compare the DTM solution (65) with the Taylor (McLaurin) expansion of the solution (34) for the same equation we noticed and found the two are similar to each other. Also, from Eqs. (72) and (71) and Figs. 3 and 4 we see the similarity of the EDAM solution and the numerical solution using DTM. The obtained solutions have a significance to explain the nature of pulse propagation in the highly non- linear medium, especially like chirping and pulse shape. Also, the similarity of the exact and numerical solution shows the efficiency and convergence of the DTM. The results obtained by ESEM, and EDAM will allow new future studies for these models and other problems.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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